

Statistical inference for moving-average Lévy-driven processes: Fourier-based approach

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Abstract

We consider a new method of the semiparametric statistical estimation for the continuous-time moving average Lévy processes. We derive the convergence rates of the proposed estimators, and show that these rates are optimal in the minimax sense.

Keywords: moving average, Lévy processes, low-frequency estimation, Fourier methods

1. Introduction

Generally speaking, continuous-time Lévy-driven moving average processes are defined as

$$Z_t = \int_{-\infty}^{\infty} \mathcal{K}(t-s) dL_s \quad (1)$$

where \mathcal{K} is a deterministic kernel and $L = (L_t)_{t \in \mathbb{R}}$ is a two-sided Lévy process with Lévy triplet (γ, σ, ν) . The conditions which guarantee that this integral is well-defined are given in the pioneering work by Rajput and Rosinski [6]. For instance, if $\int x^2 \nu(dx) < \infty$, it is sufficient to assume that $\mathcal{K} \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$. Some popular choices for the kernel are $\mathcal{K}(t) = t^\alpha e^{-\lambda t} 1_{[0, \infty)}(t)$ with $\lambda > 0$ and $\alpha > -1/2$, (Gamma-kernels, see e.g. Barndorff-Nielsen and Schmiegel [1]), or $\mathcal{K}(t) = e^{-\lambda|t|}$ (well-balanced Ornstein-Uhlenbeck process, see Schnurr and Woerner [7]).

Recently, Belomestny, Panov and Woerner [3] consider the statistical estimation of the Lévy measure ν from the low-frequency observations of the process (Z_t) . The approach presented in [3] is rather general - in particular, it works well under various choices of \mathcal{K} . Nevertheless, this approach is based on the superposition of the Mellin and Fourier transforms of the Lévy measure, and therefore its practical implementation can meet some computational difficulties.

In this paper, we present another method, which essentially uses the fact that in some cases there exists a direct relation between the characteristic exponent of the process L and the characteristic function of the process Z . Therefore, the characteristic exponent can be estimated from the observations of the process Z , and further application of the Fourier techniques from Belomestny and Reiss [2] and Panov [5] leads to the construction of a consistent estimator of the Lévy triplet.

The paper is organised as follows. In the next session, we provide the specifications of our model. In Section 3, we present the key mathematical idea, which lies in the core of the estimation procedure presented in Section 4. The upper and lower error bounds for the proposed estimates are given in Section 5. Joint consideration of the corresponding results, Theorems 1 and 2, yields the optimality of the estimates. Finally, in Section 6, we illustrate our approach with some numerical examples. All proofs are collected in Section 7.

2. Set-up

In this work, we consider the integrals of the form (1), where \mathcal{K} is a symmetric kernel of the form:

$$\mathcal{K}_\alpha(x) := (1 - \alpha|x|)^{\frac{1}{\alpha}}, \quad |x| \leq \alpha^{-1} \quad (2)$$

for some $\alpha \in (0, 1)$. As a limiting case for $\alpha \searrow 0$, we get the exponential kernel $\mathcal{K}_0(x) = \exp(-x)$. Here, for simplicity, we restrict our attention to a particular class of two-sided Lévy processes with jumps represented by a compound Poisson process CPP_t ,

$$L_t = \gamma t + \sigma W_t + CPP_t^{(1)} \cdot \mathbb{I}\{t \geq 0\} + CPP_t^{(2)} \cdot \mathbb{I}\{t < 0\}, \quad (3)$$

$$CPP_t^{(k)} := \sum_{j=1}^{N_t^{(k)}} Y_j^{(k)}, \quad k = 1, 2, \quad (4)$$

where $\gamma \in \mathbb{R}$ is a drift, $\sigma \geq 0$, W_t is a Brownian motion, $N_t^{(1)}, N_t^{(2)}$, are 2 Poisson processes with intensity λ , $Y_1^{(1)}, Y_2^{(1)}, \dots$ and $Y_1^{(2)}, Y_2^{(2)}, \dots$ are i.i.d. r.v.'s with absolutely continuous distribution, and all Y 's, $N_t^{(1)}, N_t^{(2)}, W_t$ are jointly independent. Due to the Lévy-Khintchine formula, the characteristic exponent of L is given by

$$\begin{aligned} \psi(u) = \log \mathbb{E} [e^{iuL_1}] &= i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1) \nu(dx) \\ &= i\gamma u - \frac{1}{2}\sigma^2 u^2 - \lambda + \mathcal{F}[\nu](u), \end{aligned} \quad (5)$$

where ν is a Lévy measure of (L_t) , and $\mathcal{F}[\nu](u) = \int_{\mathbb{R}} e^{iux} \nu(dx)$ stands for the Fourier transform of ν .

It is important to note that the process $(Z_t)_{t \in \mathbb{R}}$ is strictly stationary with the characteristic function of the form

$$\Phi(u) := \mathbb{E} [e^{iuZ_t}] = \exp(\Psi(u)), \quad \text{where} \quad \Psi(u) := \int_{\mathbb{R}} \psi(u \mathcal{K}(s)) ds, \quad (6)$$

and therefore for any time points t_1, \dots, t_n , the r.v.'s Z_{t_1}, \dots, Z_{t_n} are identically distributed (but dependent). Our aim is to estimate the Lévy triplet (γ, σ, ν) based on the equidistant observations of the process Z_t at the time points $\Delta, 2\Delta, \dots, n\Delta$, where $\Delta > 0$ is fixed (low-frequency set-up).

3. Main idea

The key observation is that under our choice of the kernel function \mathcal{K} , we can represent the characteristic exponent $\psi(\cdot)$ of the process (L_t) via the characteristic function $\Phi(\cdot)$ of the process Z_t . More precisely, since

$$\mathcal{K}'_{\alpha}(x) = -(1 - \alpha x)^{\frac{1-\alpha}{\alpha}} = -\mathcal{K}_{\alpha}^{1-\alpha}(x),$$

we have

$$\begin{aligned}\Phi(u) &= \exp \left[2 \int_0^{1/\alpha} \psi(u\mathcal{K}_\alpha(x)) dx \right] \\ &= \exp \left[2 \int_0^1 \psi(uy)y^{\alpha-1} dy \right] = \exp \left[2 u^{-\alpha} \int_0^u \psi(z)z^{\alpha-1} dz \right].\end{aligned}\quad (7)$$

Therefore, we derive

$$\psi(u) = \frac{1}{2}u^{1-\alpha} (u^\alpha \log(\Phi(u)))' = \frac{1}{2} \left(\alpha \log(\Phi(u)) + u \frac{\Phi'(u)}{\Phi(u)} \right), \quad (8)$$

since $\psi(u)u^{\alpha-1} \rightarrow 0$ as $u \rightarrow +0$ provided that $\int |x|\nu(dx) < \infty$, see Lemma 1. Therefore, the characteristic exponent ψ can be directly estimated from data via a plug-in estimator based on the empirical characteristic function of Z .

Moreover, returning to the representation (5), we conclude that the Lévy triplet (γ, σ, ν) can be estimated from ψ . In fact, since ν is absolutely continuous with an absolutely integrable density, then by the Riemann-Lebesgue lemma (see [4], p. 43) $\mathcal{F}[\nu](u) \rightarrow 0$ as $|u| \rightarrow \infty$, and consequently $\psi(u)$ can be viewed, at least for large $|u|$, as a second order polynomial with the coefficients $(-\lambda, i\gamma, -\sigma^2/2)$. This observation gives rise for the estimation procedure, which we present in the next session.

4. Estimation procedure

Assume that the process (Z_t) is observed on the equidistant time grid $t \in \{\Delta, 2\Delta, \dots, n\Delta\}$, where Δ is fixed.

Step 1: estimation of ψ . Define

$$\Phi_n(u) = \frac{1}{n} \sum_{j=1}^n e^{iuZ_{j\Delta}},$$

and set

$$\psi_n(u) = \frac{1}{2} \left(\alpha \log(\Phi_n(u)) + u \frac{\Phi_n'(u)}{\Phi_n(u)} \right),$$

where the branch of the complex logarithm is taken in such a way that ψ_n is continuous on $(-x_{0,n}, x_{0,n})$ with $\psi_n(0) = 0$ and $x_{0,n}$ being the first zero of

Φ_n . In fact, since Φ does not vanish on \mathbb{R} , we have $x_{0,n} \xrightarrow{a.s.} \infty$.

Step 2: estimation of σ and λ . Let $U_n \rightarrow \infty$ and

$$\tilde{w}^{U_n}(u) := (1/U_n) \tilde{w}(u/U_n),$$

where $\tilde{w}(u)$ is a continuous function, supported on the interval $[\varepsilon, 1]$ with some $\varepsilon > 0$. Consider now the optimisation problem

$$(\sigma_n^2, \lambda_n) := \operatorname{argmin}_{(\sigma^2, \lambda)} \int_0^\infty \tilde{w}^{U_n}(u) (\operatorname{Re}[\psi_n(u)] + \sigma^2 u^2/2 + \lambda)^2 du, \quad (9)$$

which has the solution

$$\sigma_n^2 = \int_0^\infty w_\sigma^{U_n}(u) \operatorname{Re} \psi_n(u) du, \quad (10)$$

with

$$w_\sigma^{U_n}(u) := \tilde{w}^{U_n}(u) \frac{2 \left[\left(\int_0^\infty \tilde{w}^{U_n}(s) ds \right) u^2 - \int_0^\infty \tilde{w}^{U_n}(s) s^2 ds \right]}{\left(\int_0^\infty \tilde{w}^{U_n}(s) s^2 ds \right)^2 - \int_0^\infty \tilde{w}^{U_n}(s) s^4 ds \cdot \int_0^\infty \tilde{w}^{U_n}(s) ds}. \quad (11)$$

Note that the weighting function $w_\sigma^{U_n}(u)$ satisfies the property $w_\sigma^{U_n}(u) = U_n^{-3} w_\sigma^1(u/U_n)$, and moreover,

$$\int_0^{U_n} (-u^2/2) w_\sigma^{U_n}(u) du = 1, \quad \int_0^{U_n} w_\sigma^{U_n}(u) du = 0. \quad (12)$$

Analogously,

$$\lambda_n = \int_0^\infty w_\lambda^{U_n}(u) \operatorname{Re} \psi_n(u) du \quad (13)$$

holds with $w_\lambda^{U_n}(u) = U_n^{-1} w_\lambda^1(u/U_n)$ satisfying the properties

$$\int_0^{U_n} (-1) w_\lambda^{U_n}(u) du = 1, \quad \int_0^{U_n} (-u^2/2) w_\lambda^{U_n}(u) du = 0.$$

Step 3: estimation of γ . Finally, the parameter γ can be estimated by considering the optimisation problem

$$\gamma_n := \operatorname{argmin}_{\gamma} \int_0^{\infty} \tilde{w}^{U_n}(u) (\operatorname{Im} \psi_n(u) - \gamma u)^2 du, \quad (14)$$

which leads to the estimate

$$\gamma_n = \int_0^{\infty} w_{\gamma}^{U_n}(u) \operatorname{Im} \psi_n(u) du, \quad (15)$$

where $w_{\gamma}^{U_n}(u) = U_n^{-2} w_{\gamma}^1(u/U_n)$ fulfills $\int_0^{U_n} u w_{\gamma}^{U_n}(u) du = 1$. All functions w_{σ}^1 , w_{γ}^1 and w_{λ}^1 are supported on $[\varepsilon, 1]$ and bounded.

Step 4: estimation of the Lévy density. Note that under our assumptions on the Lévy process (L_t) (see Section 2), the Lévy measure ν possesses a density, which we denote, with a slight abuse of notation, also by $\nu(x)$. This Lévy density can be estimated as a regularised inverse Fourier transform of the remainder:

$$\nu_n(x) := \mathcal{F}^{-1} \left[\left(\psi_n(\cdot) + \frac{\sigma_n^2}{2}(\cdot)^2 - i\gamma_n(\cdot) + \lambda_n \right) w_{\nu}(\cdot/U_n) \right] (x), \quad x \in \mathbb{R}, \quad (16)$$

where w_{ν} is a weight function supported on $[-1, 1]$. Note that $\int_{\mathbb{R}} \nu_n(x) dx = \lambda_n$, if $w_{\nu}(0) = 1$.

5. Error bounds

Theorem 1. *Consider the model (1), where \mathcal{K} is a kernel in the form (2) and (L_t) is a Lévy process in the form (3) with triplet (γ, σ, ν) . Assume that the Lévy density ν is s -times weakly differentiable for some $s \in \mathbb{N}$, and moreover the Lévy triplet belongs to the class*

$$\mathcal{T}_s = \mathcal{T}_s(\sigma^{\circ}, R) = \left\{ \sigma \in (0, \sigma^{\circ}), \quad \int x^2 \nu(dx) \leq R, \quad \|\nu^{(s)}\|_{\infty} \leq R \right\}$$

with some $\sigma^{\circ}, R > 0$. Assume also that the weighting functions satisfy the conditions

$$\|\mathcal{F}(w_{\sigma}^1(u)/u^s)\|_{L^1} < \infty, \quad \|\mathcal{F}(w_{\lambda}^1(u)/u^s)\|_{L^1} < \infty, \quad (17)$$

$$\|\mathcal{F}(w_\gamma^1(u)/u^s)\|_{L^1} < \infty. \quad (18)$$

Then it holds

$$\begin{aligned} \lim_{A \rightarrow +\infty} \overline{\lim}_{n \rightarrow +\infty} \sup_{(\gamma, \sigma, \nu) \in \mathcal{T}_s} \mathbb{P} \{ |\sigma_n^2 - \sigma^2| \geq A \cdot U_n^{-(s+3)} \} &= 0, \\ \lim_{A \rightarrow +\infty} \overline{\lim}_{n \rightarrow +\infty} \sup_{(\gamma, \sigma, \nu) \in \mathcal{T}_s} \mathbb{P} \{ |\gamma_n - \gamma| \geq A \cdot U_n^{-(s+2)} \} &= 0, \\ \lim_{A \rightarrow +\infty} \overline{\lim}_{n \rightarrow +\infty} \sup_{(\gamma, \sigma, \nu) \in \mathcal{T}_s} \mathbb{P} \{ |\lambda_n - \lambda| \geq A \cdot U_n^{-(s+1)} \} &= 0, \end{aligned}$$

provided $U_n = \sqrt{\kappa \log(n)}$ with some constant $\kappa > 0$ depending on σ° and R .

As shown in the next theorem, the above rates are optimal in minimax sense.

Theorem 2. For any $\sigma^\circ, R > 0$, there exists some $A > 0$ such that

$$\begin{aligned} \underline{\lim}_{n \rightarrow +\infty} \inf_{\check{\sigma}_n} \sup_{(\gamma, \sigma, \nu) \in \mathcal{T}_s} \mathbb{P} \{ |\check{\sigma}_n^2 - \sigma^2| \geq A \cdot (\log(n))^{-(s+3)/2} \} &> 0, \\ \underline{\lim}_{n \rightarrow +\infty} \inf_{\check{\gamma}_n} \sup_{(\gamma, \sigma, \nu) \in \mathcal{T}_s} \mathbb{P} \{ |\check{\gamma}_n - \gamma| \geq A \cdot (\log(n))^{-(s+2)/2} \} &> 0, \\ \underline{\lim}_{n \rightarrow +\infty} \inf_{\check{\lambda}_n} \sup_{(\gamma, \sigma, \nu) \in \mathcal{T}_s} \mathbb{P} \{ |\check{\lambda}_n - \lambda| \geq A \cdot (\log(n))^{-(s+1)/2} \} &> 0. \end{aligned}$$

where the infimums are taken over all possible estimates $\check{\sigma}_n, \check{\gamma}_n, \check{\lambda}_n$ of the parameters σ, γ, λ , and supremums - over all triplets from the class $\mathcal{T}_s = \mathcal{T}_s(\sigma^\circ, R)$.

6. Numerical example

Consider the integral (1) with the kernel $\mathcal{K}(x)$ from the class (2), and the Lévy process (L_t) defined by (3)-(4). For simulation study, we take $\gamma = 5$, $\lambda = 1$ and $\sigma = 0$, and aim to estimate these parameters under different choices of the parameter α , namely $\alpha = 0.5, 0.8$ and 0.9 .

Simulation. For $k = 1, 2$, denote the jump times of $L_t^{(k)}$ by $s_1^{(k)}, s_2^{(k)}, \dots$,

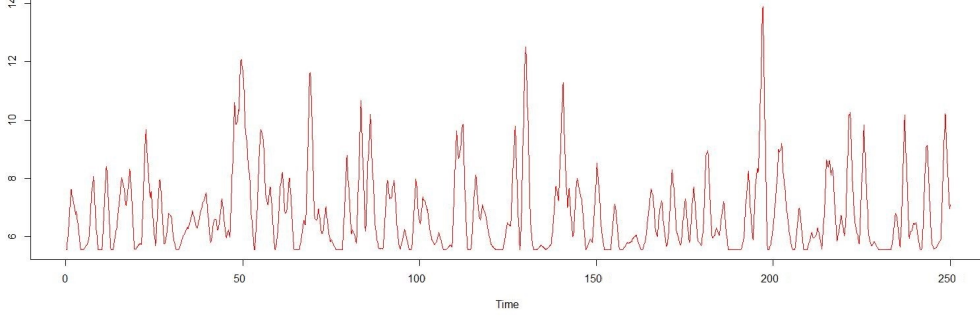


Figure 1: Typical trajectory of the process Z_t with value of the parameter $\alpha = 0.8$

corresponding to the jump sizes $Y_1^{(k)}, Y_2^{(k)}, \dots$. Note that

$$Z_t = \begin{cases} \frac{2\gamma}{1+\alpha} + \sum_{k \in K^{(1)}} \left(1 - \alpha|t - s_k^{(1)}|\right)^{1/\alpha} Y_k^{(1)}, & \text{if } t \geq \frac{1}{\alpha} \\ \frac{2\gamma}{1+\alpha} + \sum_{k \in K^{(2)}} \left(1 - \alpha|t - s_k^{(1)}|\right)^{1/\alpha} Y_k^{(1)} \\ \quad + \sum_{k \in K^{(3)}} \left(1 - \alpha|t + s_k^{(2)}|\right)^{1/\alpha} Y_k^{(2)}, & \text{if } t < \frac{1}{\alpha}, \end{cases}$$

where

$$\begin{aligned} K^{(1)} &:= \left\{ k : t - \frac{1}{\alpha} \leq s_k^{(1)} \leq t + \frac{1}{\alpha} \right\}, \\ K^{(2)} &:= \left\{ k : 0 \leq s_k^{(1)} \leq t + \frac{1}{\alpha} \right\}, \\ K^{(3)} &:= \left\{ k : 0 \leq s_k^{(2)} \leq \frac{1}{\alpha} - t \right\}. \end{aligned}$$

Typical trajectory of the process Z_t is presented on Figure 1.

Estimation. Following the ideas from Section 4, we estimate the parameters γ, λ, σ under different choices of α .

To show the convergence properties of the considered estimates, we provide simulations with different values of n . The boxplots of the corresponding estimation errors (differences) based on 25 simulation runs are presented on Figures 2, 3 and 4. Note that the parameter U_n is chosen by numerical

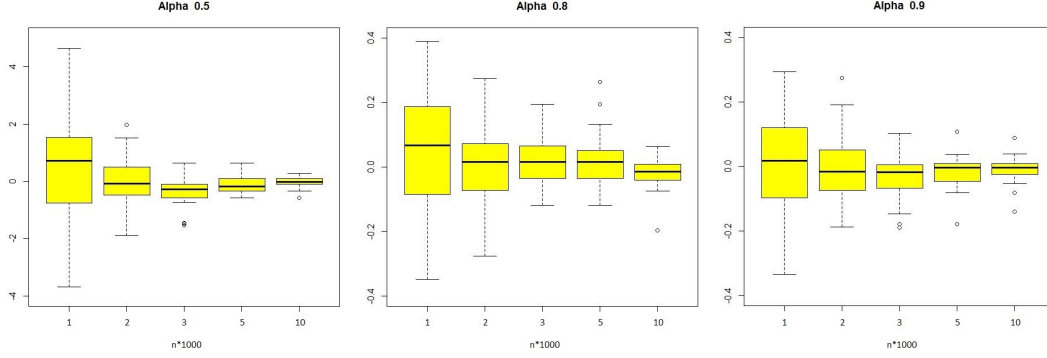


Figure 2: Boxplots $\hat{\gamma} - \gamma$ based on 25 simulation runs

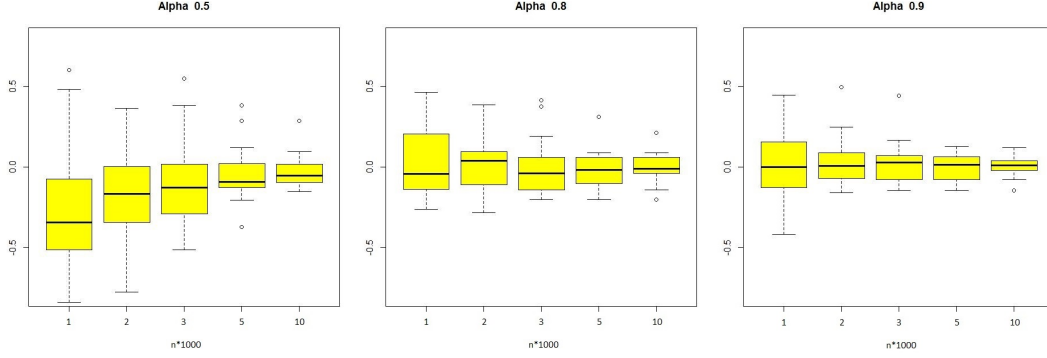


Figure 3: Boxplots of $\hat{\lambda} - \lambda$ based on 25 simulation runs

optimisation. The exact values are presented in Tables 1 and 2.

The simulation study illustrates our theoretical results on the rates of convergence given in Section 5. In fact, visual comparison of Figures 2, 3 and 4 shows that the proposed estimator for the parameter σ has the highest speed of convergence to the true value, whereas the corresponding speed for γ_n is lower, and for λ_n even more low (cf with the rates in Theorem 1). Moreover, the simulations results show that the convergence rates significantly depend on the parameter α . More precisely, it turns out that the quality of estimation increases with growing α , and the best rates correspond to the case when α is close to 1. This can be explained by the fact that observations become less independent as α increases.

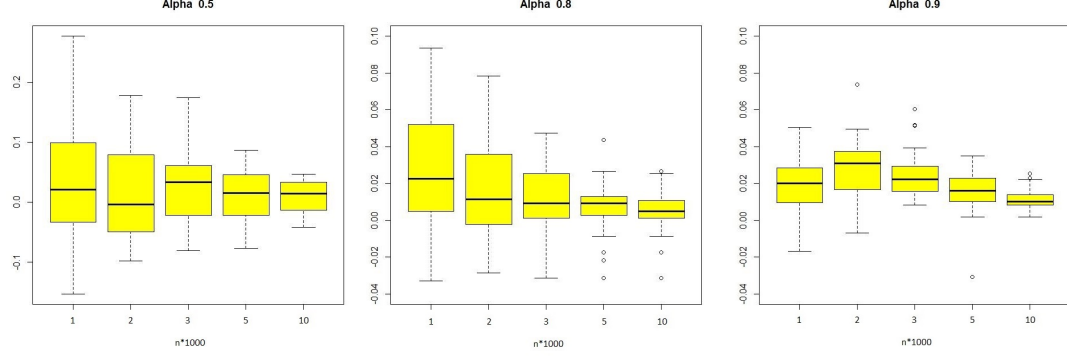


Figure 4: Boxplots of $\hat{\sigma}^2 - \sigma^2$ based on 25 simulation runs

$\alpha = 0.5$		$\alpha = 0.8$		$\alpha = 0.9$	
n	U_n	n	U_n	n	U_n
1000	1.2	1000	3.5	1000	4.5
2000	1.35	2000	3.5	2000	4.5
3000	1.4	3000	3.8	3000	4.6
5000	1.45	5000	4	5000	4.8
10000	1.55	10000	4.2	10000	5.1

Table 1: Optimal sequences for the estimation of parameter λ

$\alpha = 0.5$		$\alpha = 0.8$		$\alpha = 0.9$	
n	U_n	n	U_n	n	U_n
1000	0.8	1000	2.7	1000	3
2000	0.85	2000	2.75	2000	3.2
3000	1	3000	2.75	3000	3.2
5000	1.1	5000	2.8	5000	3.3
10000	1.3	10000	3	10000	3.5

Table 2: Optimal sequences for the estimation of parameter γ

$\alpha = 0.5$		$\alpha = 0.8$		$\alpha = 0.9$	
n	U_n	n	U_n	n	U_n
1000	8	1000	8.5	1000	8.75
2000	8	2000	8.55	2000	8.8
3000	8	3000	8.6	3000	8.8
5000	8.2	5000	8.7	5000	9
10000	8.5	10000	8.8	10000	9.2

Table 3: Optimal sequences for the estimation of parameter σ

7. Proofs

7.1. Proof of Theorem 1

1. For the sake of clarity we focus our analysis on the estimate σ_n . First note that by (10) and (12) the difference $\sigma_n^2 - \sigma^2$ can be decomposed as follows:

$$\begin{aligned}
\sigma_n^2 - \sigma^2 &= \int_0^{U_n} w_\sigma^{U_n}(u) \operatorname{Re}(\psi_n(u) - \psi(u)) \, du \\
&\quad + \int_0^{U_n} w_\sigma^{U_n}(u) \operatorname{Re} \psi(u) \, du - \sigma^2 \\
&= \underbrace{\int_0^{U_n} w_\sigma^{U_n}(u) \operatorname{Re}(\psi_n(u) - \psi(u)) \, du}_{\text{Statistical error}} + \underbrace{\int_0^{U_n} w_\sigma^{U_n}(u) \operatorname{Re} \mathcal{F}[\nu](u) \, du}_{\text{Bias}}.
\end{aligned} \tag{19}$$

2. Let us first consider the bias term in (19). Note that its order obviously depends on the decay of the Fourier transform $\mathcal{F}[\nu](u)$, which is related to

the smoothness of ν , see [4]. Then by the Plancherel identity

$$\begin{aligned}
\left| \int_0^{U_n} w_\sigma^{U_n}(u) \operatorname{Re} \mathcal{F}[\nu](u) du \right| &\leq \left| \int_0^\infty w_\sigma^{U_n}(u) \mathcal{F}[\nu](u) du \right| \\
&= 2\pi \left| \int_{-\infty}^\infty \nu^{(s)}(x) \overline{\mathcal{F}^{-1}[w_\sigma^{U_n}(\cdot)/(i\cdot)^s](x)} dx \right| \\
&\leq U_n^{-(s+3)} \|\nu^{(s)}\|_\infty \|\mathcal{F}(w_\sigma^1(u)/u^s)\|_{L^1} \lesssim U_n^{-(s+3)},
\end{aligned} \tag{20}$$

since $w_\sigma^{U_n}(u) = U_n^{-3} w_\sigma^1(u/U_n)$, $\|\nu^{(s)}\|_\infty \leq R$, and (17).

2. As for the statistical error, we first note that

$$\psi_n(u) - \psi(u) = \frac{\alpha}{2} (\log(\Phi_n(u)) - \log(\Phi(u))) + \frac{u}{2} \left(\frac{\Phi'_n(u)}{\Phi_n(u)} - \frac{\Phi'(u)}{\Phi(u)} \right).$$

Consider the event

$$\mathcal{B}_{n,A} := \left\{ \max_{j=0,1} \sup_{u \in [-U_n, U_n]} |D_{n,j}(u)| \leq A\varepsilon_n \right\},$$

where $D_{n,j}(u) = (\Phi_n^{(j)}(u) - \Phi^{(j)}(u))/\Phi(u)$, $j = 0, 1$, and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Using the same techniques as in Theorem 2 from [3], one can show that from the condition $\int_{|x|>1} x^2 \nu(dx) < \infty$, it follows that

$$\mathbb{P}\{\mathcal{B}_{n,A}\} \geq 1 - \frac{C_1}{\sqrt{A}} \frac{\sqrt{U_n} n^{(1/4)-C_1 A^2}}{\log^{1/4}(n)},$$

provided $\varepsilon_n = \sqrt{\log(n)/n} \exp\{C_2 \sigma^2 U_n^2\}$ with some $C_1, C_2 > 0$ depending on α . On the event $\mathcal{B}_{n,A}$, it holds

$$\log(\Phi_n(u)) - \log(\Phi(u)) = D_{n,0}(u) + O(|D_{n,0}(u)|^2),$$

because $|\log(1+z) - z| \leq 2|z|^2$ for any $|z| < 1/2$. Moreover,

$$\begin{aligned}
\frac{\Phi'_n(u)}{\Phi_n(u)} - \frac{\Phi'(u)}{\Phi(u)} &= \frac{\Phi(u)\Phi'_n(u) - \Phi'(u)\Phi_n(u)}{\Phi_n(u)\Phi(u)} \\
&= \frac{(\Phi(u) - \Phi_n(u))\Phi'_n(u) + (\Phi'_n(u) - \Phi'(u))\Phi_n(u)}{\Phi_n(u)\Phi(u)} \\
&= -\frac{\Phi'_n(u)}{\Phi_n(u)}D_{n,0}(u) + D_{n,1}(u) \\
&= \left[\frac{\Phi'(u)}{\Phi(u)} - \frac{\Phi'_n(u)}{\Phi_n(u)} \right] D_{n,0}(u) - \frac{\Phi'(u)}{\Phi(u)}D_{n,0}(u) + D_{n,1}(u),
\end{aligned}$$

and therefore on the event \mathcal{A} ,

$$\begin{aligned}
\frac{\Phi'_n(u)}{\Phi_n(u)} - \frac{\Phi'(u)}{\Phi(u)} &= \frac{D_{n,1}(u) - \frac{\Phi'(u)}{\Phi(u)}D_{n,0}(u)}{1 + D_{n,0}(u)} \\
&= D_{n,1}(u) - \frac{\Phi'(u)}{\Phi(u)}D_{n,0}(u) \\
&\quad + O(|D_{n,0}(u)|^2) + O(|D_{n,0}(u)D_{n,1}(u)|),
\end{aligned}$$

where we use the inequality $|(1+z)^{-1} - 1| \leq 2|z|$ for any $|z| < 1/2$. Therefore, the statistical error can be further decomposed as follows:

$$\int_0^{U_n} w_\sigma^{U_n}(u) \operatorname{Re}(\psi_n(u) - \psi(u)) du = \frac{1}{2} (L_n + R_n)$$

with the first order (linear) term $L_n = \operatorname{Re} \check{L}_n$,

$$\begin{aligned}
\check{L}_n &:= \int_0^{U_n} w_\sigma^{U_n}(u) \left(\alpha - u \frac{\Phi'(u)}{\Phi(u)} \right) D_{n,0}(u) du \\
&\quad + \int_0^{U_n} w_\sigma^{U_n}(u) u D_{n,1}(u) du
\end{aligned}$$

and the remainder R_n , which contains higher order powers of $D_{n,0}$ and $D_{n,1}$. On the event $\mathcal{B}_{n,A}$,

$$|R_n| \leq \max \left(\max_u |D_{n,0}|^2, \max_u |D_{n,0}(u)D_{n,1}(u)| \right) \int_0^{U_n} w_\sigma^{U_n}(u) du \lesssim A^2 \frac{\varepsilon_n^2}{U_n^2},$$

and we finally conclude that at least for large n it holds

$$\mathbb{P} \{ |R_n| > A^2 g_{n,1} \} \leq \frac{C_1}{\sqrt{A}} \frac{\sqrt{U_n} n^{(1/4) - C_1 A^2}}{\log^{1/4}(n)}, \quad (21)$$

where

$$g_{n,1} = C_3 \frac{\log(n)}{n} \frac{e^{2C_2 \sigma^2 U_n^2}}{U_n^2}$$

with some $C_3 > 0$.

4. The linear term L_n can be analysed as follows. We have $\mathbb{E}[\check{L}_n] = 0$, $\text{Var } L_n \leq \text{Var } \check{L}_n$, and

$$\begin{aligned} \text{Var } \check{L}_n &= \int_0^{U_n} \int_0^{U_n} w_\sigma^{U_n}(u) w_\sigma^{U_n}(v) \left(\alpha - u \frac{\Phi'(u)}{\Phi(u)} \right) \overline{\left(\alpha - v \frac{\Phi'(v)}{\Phi(v)} \right)} \frac{1}{\Phi(u) \overline{\Phi(v)}} \\ &\quad \cdot \text{cov}_{\mathbb{C}}(\Phi_n(u), \Phi_n(v)) \, du \, dv \\ &\quad + \int_0^{U_n} \int_0^{U_n} w_\sigma^{U_n}(u) w_\sigma^{U_n}(v) \left(\alpha - u \frac{\Phi'(u)}{\Phi(u)} \right) \frac{v}{\Phi(u) \overline{\Phi(v)}} \\ &\quad \cdot \text{cov}_{\mathbb{C}}(\Phi_n(u), \Phi'_n(v)) \, du \, dv \\ &\quad + \int_0^{U_n} \int_0^{U_n} w_\sigma^{U_n}(u) w_\sigma^{U_n}(v) \frac{u}{\Phi(u) \overline{\Phi(v)}} \overline{\left(\alpha - v \frac{\Phi'(v)}{\Phi(v)} \right)} \\ &\quad \cdot \text{cov}_{\mathbb{C}}(\Phi'_n(u), \Phi_n(v)) \, du \, dv \\ &\quad + \int_0^{U_n} \int_0^{U_n} w_\sigma^{U_n}(u) w_\sigma^{U_n}(v) \frac{uv}{\Phi(u) \overline{\Phi(v)}} \text{cov}_{\mathbb{C}}(\Phi'_n(u), \Phi'_n(v)) \, du \, dv \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

It holds

$$\text{cov}_{\mathbb{C}}(\Phi_n(u), \Phi_n(v)) = \frac{1}{n^2} \sum_{j,k=1}^n \text{Cov}_{\mathbb{C}}(e^{iuZ_{j\Delta}}, e^{ivZ_{k\Delta}}).$$

Let $t > s$ and compute

$$\begin{aligned}
\text{Cov}_{\mathbb{C}} [e^{iuZ_t}, e^{ivZ_s}] &= \mathbb{E} [e^{iuZ_t - ivZ_s}] - \mathbb{E} [e^{iuZ_t}] \mathbb{E} [e^{-ivZ_s}] \\
&= \Phi_{(Z_t, Z_s)}(u, -v) - \Phi_{Z_t}(u) \Phi_{Z_s}(-v) \\
&= \exp \left(\int_{\mathbb{R}} \psi(u\mathcal{K}_{\alpha}(x-t) - v\mathcal{K}_{\alpha}(x-s)) dx \right) \\
&\quad - \exp \left(\int_{\mathbb{R}} \psi(u\mathcal{K}_{\alpha}(x-t)) dx \right) \\
&\quad \times \exp \left(\int_{\mathbb{R}} \psi(-v\mathcal{K}_{\alpha}(x-s)) dx \right).
\end{aligned}$$

Using the inequality $|e^z - e^y| \leq (|e^z| \vee |e^y|) |y - z|$, which holds for any $z, y \in \mathbb{C}$, we get

$$\begin{aligned}
|\text{Cov}_{\mathbb{C}} [e^{iuZ_t}, e^{ivZ_s}]| &\leq (|\Phi_{(Z_t, Z_s)}(u, -v)| \vee |\Phi_{Z_t}(u) \Phi_{Z_s}(-v)|) \\
&\quad \times \left| \int_{\mathbb{R}} [\psi(u\mathcal{K}_{\alpha}(x-t) - v\mathcal{K}_{\alpha}(x-s)) \right. \\
&\quad \left. - \psi(u\mathcal{K}_{\alpha}(x-t)) - \psi(-v\mathcal{K}_{\alpha}(x-s))] dx \right|.
\end{aligned}$$

Due to the Lévy-Khintchine formula (5), we derive for any $u_1, u_2 \in \mathbb{R}$,

$$\begin{aligned}
|\psi(u_1 + u_2) - \psi(u_1) - \psi(u_2)| \\
= |\sigma^2 u_1 u_2| + \int_0^{\infty} |\exp(iu_1 x) - 1| |\exp(iu_2 x) - 1| \nu(dx) \leq C |u_1| |u_2|
\end{aligned}$$

with $C = \sigma^2 + \int_{\mathbb{R}} x^2 \nu(dx) < \infty$. As a result

$$\begin{aligned}
|\text{Cov}_{\mathbb{C}} [e^{iuZ_t}, e^{ivZ_s}]| \\
\leq C |uv| (|\Phi_{(Z_t, Z_s)}(u, -v)| \vee |\Phi_{Z_t}(u) \Phi_{Z_s}(-v)|) \\
\quad \times \int \mathcal{K}_{\alpha}(x) \mathcal{K}_{\alpha}(x+t-s) dx.
\end{aligned}$$

Hence

$$\begin{aligned}
|\text{Cov}_{\mathbb{C}} [\Phi_n(u), \Phi_n(v)]| &= \frac{1}{n^2} \sum_{j,k=1}^n |\text{Cov}_{\mathbb{C}} (e^{iuZ_{j\Delta}}, e^{ivZ_{k\Delta}})| \\
&\leq \frac{|uv|}{n^2} \sum_{j,k=1}^n (|\Phi_{(Z_{j\Delta}, Z_{k\Delta})}(u, -v)| \vee |\Phi(u)\Phi(-v)|) \\
&\quad \times \int \mathcal{K}_{\alpha}(x) \mathcal{K}_{\alpha}(x + (k-j)\Delta) dx,
\end{aligned}$$

where

$$\int \mathcal{K}_{\alpha}(x) \mathcal{K}_{\alpha}(x + h) dx = 0$$

if $|h| > 2/\alpha$ and

$$\int \mathcal{K}_{\alpha}(x) \mathcal{K}_{\alpha}(x + h) dx \leq 2(1 - \alpha|h|/2)^{2/\alpha}$$

for $|h| \leq 2/\alpha$. In the limiting case $\alpha \searrow 0$, we get

$$\int \mathcal{K}_{\alpha}(x) \mathcal{K}_{\alpha}(x + h) dx \leq 2e^{-|h|}.$$

As a result

$$\begin{aligned}
|\text{Cov}_{\mathbb{C}} [\Phi_n(u), \Phi_n(v)]| &\leq \frac{Q(u, v)}{n^2} \sum_{0 \leq |j-k| \leq \frac{2}{\Delta\alpha}} (1 - \alpha\Delta|j-k|/2)^{2/\alpha} \\
&\leq \frac{Q(u, v)}{n} \int_0^{\min\{n, \frac{2}{\Delta\alpha}\}} (1 - \alpha\Delta h/2)^{2/\alpha} dh \\
&= \frac{Q(u, v)}{\Delta n} \int_0^{\min\{\Delta n, \frac{2}{\alpha}\}} (1 - \alpha r/2)^{2/\alpha} dr \\
&= \frac{Q(u, v)}{\Delta n} \frac{2}{\alpha + 2} \left(1 - \min \left\{ \frac{\alpha\Delta n}{2}, 1 \right\} \right)^{2/\alpha+1},
\end{aligned}$$

where the function $Q(u, v)$ is bounded provided $\sigma > 0$.

For further analysis of the terms $I_1 - I_4$ in (22), we need also some asymptotic upper bound for the relation $|\Phi'(u)/\Phi(u)|$ for large u . Combining (7)

with (8), we get

$$\frac{\Phi'(u)}{\Phi(u)} = 2 \left(-\alpha \int_0^1 \psi(yu) y^{\alpha-1} dy + \psi(u) \right) / u,$$

and therefore it holds $|\Phi'(u)/\Phi(u)| \lesssim u$ as $|u| \rightarrow \infty$.

So we have for I_1

$$\begin{aligned} I_1 &\leq \frac{Q^*}{n} \left[\int_0^{U_n} |w_\sigma^{U_n}(u)| \cdot \left| \alpha - u \frac{\Phi'(u)}{\Phi(u)} \right| \cdot \left| \frac{1}{\Phi(u)} \right| du \right]^2 \\ &\leq \frac{Q^*}{n} \left[\int_0^{U_n} |U_n^{-3} w_\sigma^1(u/U_n)| \left(\alpha + u \left| \frac{\Phi'(u)}{\Phi(u)} \right| \right) \left| \frac{1}{\Phi(u)} \right| du \right]^2 \\ &\leq \frac{Q^*}{nU_n} \left[\int_\varepsilon^1 |w_\sigma^1(u)u| \cdot \left| \frac{\Phi'(uU_n)}{\Phi(uU_n)} \right| \cdot \left| \frac{1}{\Phi(uU_n)} \right| du \right]^2 \\ &\lesssim \frac{1}{n |\Phi(U_n)|^2} \end{aligned}$$

with some $Q^* > 0$. Analogously we get the upper bounds for I_2, I_3, I_4 , for instance,

$$I_4 \leq \frac{Q^*}{n} \left[\int_0^{U_n} |w_\sigma^{U_n}(u)u| \left| \frac{1}{\Phi(u)} \right| du \right]^2 \lesssim \frac{1}{nU_n^2 |\Phi(U_n)|^2}, \quad n \rightarrow \infty.$$

Finally, taking into account that

$$\begin{aligned} |\Phi(U_n)| &= \exp \left\{ 2 \int_0^1 \operatorname{Re}[\psi(U_n y)] y^{\alpha-1} dy \right\} \\ &= \exp \left\{ -\frac{\sigma^2 U_n^2}{2 + \alpha} - \frac{2\lambda}{\alpha} + 2 \int_0^1 \operatorname{Re} \mathcal{F}[\nu](U_n y) y^{\alpha-1} dy \right\} \gtrsim e^{-C_4 U_n^2} \end{aligned}$$

with any $C_4 > \sigma^2/(4 + 2\alpha)$, we conclude that due to Markov inequality,

$$\mathbb{P} \{ |L_n| > A g_{n,2} \} \leq \frac{1}{A^2}, \quad \text{where } g_{n,2} = \frac{e^{(C_4/2)U_n^2}}{n^{1/2}}. \quad (22)$$

4. Joint consideration of (20), (21) and (22) concludes the proof. In fact, under the choice $U_n = \sqrt{\kappa \log(n)}$ with $\kappa < \min(C_4^{-1}, (2C_2\sigma^2)^{-1})$ we get that both $g_{n,1}$ and $g_{n,2}$ are of the polynomial order.

7.2. Proof of Theorem 2

Below we focus on the proof of the first statement.

A scheme for the proof of lower bounds is introduced in [2] and (more generally) in [8]. Shortly speaking, it is sufficient to construct two models from the class $\mathcal{T}_s(\sigma^\circ, R)$, say $(\gamma_0, \sigma_0, \nu_0)$ and $(\gamma_1, \sigma_1, \nu_1)$ (depending on n), such that

$$|\sigma_1^2 - \sigma_0^2| \geq 2A \log(n)^{-(s+3)/2},$$

and the χ^2 -difference between the corresponding probability measures is bounded by $1/n$:

$$\chi^2(p_1|p_0) = \int \frac{(p_1(x) - p_0(x))^2}{p_0(x)} dx \lesssim n^{-1},$$

where p_0 and p_1 are the probability densities for the first and the second models resp.

1. Let us first present the models. The first model has the triplet $(0, \sigma_0, \nu_0) \in \mathcal{T}_s(\sigma^\circ, R)$ with $\sigma_0 = \sigma = \sigma^\circ/2$ and a Lévy density $\nu_0(x) = \nu(x) = c(1 + |x|)^{-4}$, where $c > 0$ is chosen to guarantee $\|\nu^{(s)}\|_\infty \leq R$. We now perturb (σ, ν) such that for low frequencies the characteristic functions still coincide. For this reason, we take a flat-top kernel K such that

$$\mathcal{F}K(u) = \begin{cases} 1, & |u| \leq 1, \\ \exp\left(-\frac{e^{-1/(|u|-1)}}{2-|u|}\right), & 1 < |u| < 2, \\ 0, & |u| \geq 2. \end{cases}$$

This kernel and its derivatives have polynomial decay of any order, that is, for any $r = 0, 1, 2, \dots$ and any $q = 1, 2, \dots$ it holds $K^{(r)}(x) \leq (1 + |x|)^{-q}$ at least for large $|x|$. Introduce $K_h(x) = h^{-1}K(h^{-1}x)$ for some (bandwidth) $h > 0$.

Introduce the second model via the triplet $(0, \sigma_1, \nu_1)$, where

$$\sigma_1^2 = \sigma^2 + 2\delta, \quad \nu_1(x) = \nu(x) + \delta K_h''(x)$$

with some $\delta > 0$, which we will specify latter. Note that this model also belongs to the considered class $\mathcal{T}_s(\sigma^\circ, R)$ when h is small enough, provided $\delta = o(h^3)$ since then as $h \rightarrow 0$

$$\delta |K_h''(x)| = \delta h^{-3} |K''(x/h)| \lesssim \delta h^{-3} (1 + |x|/h)^{-4} = o((1 + |x|)^{-4}) = o(\nu(x))$$

(uniformly over $x \in \mathbb{R}$) follows from the polynomial decay of K'' of any order.

2. On the second step, we consider the difference between the models. For the corresponding characteristic exponents we obtain (note $\mathcal{F}K_h''(u) = -u^2\mathcal{F}K_h(u)$, $\int K_h''(u)du = 0$):

$$\psi_1(u) - \psi_0(u) = \delta u^2(1 - \mathcal{F}K(hu)),$$

which is zero for $u \in [-h^{-1}, h^{-1}]$.

For further analysis, we need a lower bound for the marginal density p_0 of the process

$$Z_{0,s} = \int \mathcal{K}_\alpha(s-t) dL_{0,t},$$

where $L_{0,t}$ is a Levy process with triplet $(0, \sigma_0, \nu_0)$. Note that since the process Z_s is stationary, we can take any s , in particular, $s = 0$. Taking into account the decomposition (3), we conclude that

$$p_0(x) = \left(N(0, \sigma_K^2) * \sum_{k=1}^{\infty} \frac{e^{-\lambda\Delta}(\lambda\Delta)^k}{k!} q_k \right)(x),$$

where $\sigma_K^2 = \sigma_0^2 \int \mathcal{K}_\alpha^2(s) ds$ and q_k is the density of a random variable

$$\sum_{j=1}^k \mathcal{K}_\alpha(-U_j) \xi_j$$

with ξ_1, \dots, ξ_k being i.i.d random variables with density ν_0/λ and $U_1 < \dots < U_k$ being i.i.d. random variables with uniform law on $[-1/\alpha, 1/\alpha]$. In view of the positivity of the summands, $\nu_0 \gtrsim \nu$ and the exponential decay of the Gaussian density (uniformly for $\Delta \lesssim 1$ and keeping λ, σ_0, ν_0 fixed), we derive

$$p_0(x) \geq \lambda\Delta e^{-\lambda\Delta} (N(0, \sigma_K^2) * q_1)(x) \gtrsim (1 + |x|)^{-4}.$$

This yields the following upper bound for the χ^2 - difference between the models:

$$\chi^2(p_1|p_0) = \int \frac{(p_1(x) - p_0(x))^2}{p_0(x)} dx \lesssim \int (1 + |x|^4) (p_1(x) - p_0(x))^2 dx,$$

and due to the Plancherel identity, we get

$$\chi^2(p_1|p_0) \lesssim \|\Phi_1 - \Phi_0\|_{\mathcal{L}^2}^2 + \|(\Phi_1 - \Phi_0)''\|_{\mathcal{L}^2}^2 \quad (23)$$

With the inequality $|1 - e^{-z}| \leq 2|z|$ for $z = x + iy \in \mathbb{C}$ with $x \geq 0$ we can estimate the \mathcal{L}^2 -norm between the characteristic functions Φ_0 and Φ_1 :

$$\begin{aligned} \|\Phi_1 - \Phi_0\|_{L^2}^2 &\leq \int 2 \max(|\Phi_0(u)|, |\Phi_1(u)|)^2 |\Psi_1(u) - \Psi_0(u)|^2 du \\ &\lesssim \int_{|u|>h^{-1}} e^{-(\sigma^2/(\alpha+2))u^2} \left| u^{-\alpha} \int_0^u (\psi_1(z) - \psi_0(z)) z^{\alpha-1} dz \right|^2 du \\ &\lesssim \delta^2 \int_{|u|>h^{-1}} e^{-(\sigma^2/(\alpha+2))u^2} u^{-2\alpha} \left| \int_0^u (-1 + \mathcal{FK}(hz)) z^{\alpha+1} dz \right|^2 du \\ &\lesssim \delta^2 h^{-2(\alpha+2)} \int_{|u|>h^{-1}} e^{-(\sigma^2/(\alpha+2))u^2} u^{-2\alpha} \\ &\quad \times \left| \int_0^{uh} (-1 + \mathcal{FK}(y)) y^{\alpha+1} dy \right|^2 du \\ &\lesssim \delta^2 h^{-5} \int_{|v|>1} e^{-(\sigma^2/(\alpha+2))(v/h)^2} v^{-2\alpha} \\ &\quad \times \left| \int_0^v (-1 + \mathcal{FK}(y)) y^{\alpha+1} dy \right|^2 dv \\ &\lesssim \delta^2 h^{-5} e^{-(\sigma^2/(\alpha+2))(1/h)^2}. \end{aligned}$$

where we use that

$$\begin{aligned} |\Phi(u)| &= \exp \left\{ 2u^{-\alpha} \int_0^u \left(-\frac{1}{2}\sigma^2 z^2 - \lambda + \operatorname{Re} \mathcal{F}[\nu](z) \right) z^{\alpha-1} dz \right\} \\ &\leq \exp \left\{ -(2\sigma^2/(2\alpha+4))u^2 \right\}, \end{aligned}$$

since $|\operatorname{Re} \mathcal{F}[\nu](z)| < \lambda$. Analogously, we get the upper bound for the second

summand in (23):

$$\begin{aligned}
& \|(\Phi_1 - \Phi_0)''\|_{L^2}^2 \\
&= \int_{|u|>h^{-1}} \left| (\Psi_1''(u) + \Psi_1'(u)) \Phi_1(u) + (\Psi_0''(u) + \Psi_0'(u)) \Phi_0(u) \right|^2 du \\
&\lesssim \int_{|u|>h^{-1}} \left| (\Psi_1''(u) + \Psi_1'(u)) + (\Psi_0''(u) + \Psi_0'(u)) \right|^2 e^{-(2\sigma^2/(\alpha+2))u^2} du.
\end{aligned}$$

Due to the definition of the class $(\int x^2 \nu(dx) < \infty)$ and to the assumption $\nu(\mathbb{R}) < \infty$, we get that $|\psi'(u)| \lesssim 1 + |u|$ and $|\psi''(u)| \lesssim 1$ as $u \rightarrow \infty$. Therefore, applying (6), we get the same asymptotics for the first and second derivatives of the function $\Psi_0(u)$, whereas $\Psi_1'(u) \lesssim 1 + |u| + \delta h^{-1}$ and $\Psi_1''(u) \lesssim 1 + \delta h^{-2}$. Finally, we get

$$\|(\Phi_1 - \Phi_0)''\|_{L^2}^2 \lesssim \delta^2 h^{-4} e^{-(2\sigma^2/(\alpha+2))(1/h)^2}.$$

3. To conclude the proof, we choose $\delta = \delta' h^{s+3}$ with fixed δ' , and

$$h = (2\sigma^2/(\alpha+2))^{1/2} (\log(n))^{-1/2}.$$

Then

$$\chi^2(p_1|p_0) \lesssim h^{2s+1} e^{-(2\sigma^2/(\alpha+2))(1/h)^2} \lesssim n^{-1},$$

and

$$|\sigma_1^2 - \sigma_0^2| = 2\delta = C (\log(n))^{-(s+3)/2}$$

with some constant C depending on α and σ° . This observation completes the proof.

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Appendix. Some auxiliary results

Lemma 1. *Let $\psi(u)$ be a characteristic exponent of L_t in the form (5), and let $\int |x|\nu(dx) < \infty$. Then for any $\alpha > 0$, it holds $\lim_{u \rightarrow 0+} \psi(u)u^{\alpha-1} = 0$.*

Proof. Note that

$$\lim_{u \rightarrow 0+} \frac{\psi(u)}{u^{1-\alpha}} = \lim_{u \rightarrow 0+} \frac{i\gamma u - (\sigma^2/2)u^2}{u^{1-\alpha}} + \lim_{u \rightarrow 0+} \left[u^\alpha \cdot \int_{\mathbb{R}/\{0\}} \frac{e^{iux} - 1}{u} \nu(dx) \right] = 0,$$

since

$$\lim_{u \rightarrow 0+} \int_{\mathbb{R}/\{0\}} \frac{e^{iux} - 1}{u} \nu(dx) = \int_{\mathbb{R}/\{0\}} \lim_{u \rightarrow 0+} \frac{e^{iux} - 1}{u} \nu(dx) = i \int_{\mathbb{R}/\{0\}} x \nu(dx),$$

where the change of places between limit and integral is possible due to the Lebesgue theorem. In fact,

$$\left| \frac{e^{iux} - 1}{u} \right| \leq \frac{1 - \cos(ux)}{u} + \frac{|\sin(ux)|}{u} \leq 2|x|,$$

and $\int |x|\nu(dx) < \infty$ due to the assumption. \square

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